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Norms of Analytic Interpolation Projections on General Domains

J. G. CLUNIE

Department of Mathematics, The Open University, Milton Keynes, Bedfordshire, England

AND

J. C. MASON

Mathematics Branch, Royal Military College of Science, Shrivenham, Swindon, Wiltshire, England

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For any finitely connected open domain D bounded by Jordan curves, it is proved that there exists a function in $A(D)$, the space of functions analytic in D and continuous on $\bar{D}$, having a maximum modulus of at most unity and taking given values of modulus at most unity at a finite number of given points on ∂D. Explicit constructions of such functions are given for the special cases of a disc, a domain between two circles, an elliptical ring, and an ellipse with a slit between its foci. It is hence proved that the projection I_n from $A(D)$ onto the subspace of polynomials of fixed degree $n-1$ which interpolate in n given points of ∂D, has a Chebyshev norm equal to the supremum of the sum of the absolute values of the fundamental polynomials in the Lagrange interpolation formula. Similar results are also proved in the special cases of a domain between two circles with centres ζ_0, ζ_1 and an ellipse with a slit between its foci ± 1, when the interpolation subspaces are more appropriately chosen to be, respectively, polynomials of degrees n in $(z-\zeta_1)^{-1}$ and $m-1$ in $z-\zeta_0$, and functions of the form $A_n(z)+\sqrt{(z^2-1)}\,B_{n-1}(z)$, where A_n and B_{n-1} are polynomials of respective degrees n and $n-1$.

1. INTRODUCTION

Consider the space $A(D)$ of functions analytic in a finitely connected open domain D and continuous on $\bar{D}$, where the boundary ∂D consists of Jordan curves. Suppose that $f(z) \in A(D)$ is interpolated by a polynomial $I_n f$ of degree $n-1$ in z at n distinct points z_k $(k=1,\ldots,n)$ of ∂D, where I_n denotes the projection operator. Let $\mathbb{C}$, $\mathbb{N}$ denote the complex plane and the positive integers, respectively.

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Then it is well known (see [1]) that

$$\|f - I_n f\| \leqslant (1 + \|I_n\|)\, \|f - f_n^B\|,$$

where

$$\|f\| = \|f\|_\infty = \max_{\bar{D}} |f| = \max_{\partial D} |f|,$$

and f_n^B is a minimax polynomial approximation of degree $n-1$. Hence, $\|I_n\|$ bounds the relative closeness of $I_n f$ to a minimax approximation, and so its numerical value is of practical importance, as also are its properties as a function of $\{z_k\}$. Now

$$(I_n f)(z) = \sum_{k=1}^{n} f(z_k)\, l_k(z), \tag{1}$$

where $l_k(z)$ are the fundamental polynomials of the Lagrange interpolation formula

$$l_k(z) = \prod_{j \neq k} \left(\frac{z - z_j}{z_k - z_j} \right). \tag{2}$$

Assuming that attention is restricted to functions for which $\|f\| = 1$, we deduce from (1) that

$$\|I_n f\| \leqslant \gamma_n$$

and, hence,

$$\|I_n\| \leqslant \gamma_n, \tag{3}$$

where

$$\gamma_n = \sup_z \sum |l_k(z)|. \tag{4}$$

If z^* is a point of attainment of this supremum and if there exists an analytic function $f(z)$, with $\|f\| = 1$, such that

$$f(z_k) = \sigma_k = \operatorname{sgn} l_k(z^*) \qquad (k = 1, \ldots, n), \tag{5}$$

where

$$\operatorname{sgn} z = z/|z|,$$

then it follows from (1) that

$$(I_n f)(z^*) = \gamma_n,$$

and, hence, that

$$\|I_n\| = \gamma_n. \tag{6}$$

The formula (4) for γ_n is computationally convenient and, indeed, has been used successfully, for example, by Geddes [2] on the ellipse to provide numerical bounds via (3) for $\|I_n\|$ and, hence, also for $\|f - f_n^B\|$. However, it follows from (6) that γ_n is actually equal to (rather than just a bound for) $\|I_n\|$, and so the theory is tightened, provided that an analytic function $f(z)$ of unit norm does indeed exist which satisfies (5).

In deriving the formula (6) here, we are following the logic used in establishing the same formula (in x) for interpolation on the real line (see, e.g., [1]), which depends on the existence of a continuous function of norm unity taking values ± 1 at given points. However, while the latter may be constructed by simply joining up the values ± 1 by straight lines, there is no such trivial realisation of an analytic function with the required properties in the complex plane. In this paper we construct such a function for any n-tuply connected bounded domain in $\mathbb{C}$ bounded by n Jordan curves. We first deal with such domains when the bounding Jordan curves are circles and then extend these results to the general situation by an appropriate appeal to well-known results on conformal mapping.

Before proceeding further, we should make due reference to some earlier related work. In the case of the unit disc, a theorem of Rudin [3] establishes the existence of an analytic function of unit norm with the sign property (5), and more recently in an unpublished report Brutman [4] has given a construction of such a function based on a number of compositions and an appeal to the Riemann mapping theorem. Our corresponding construction in Theorem 2.2 and Corollary 3.1 is more general (for a domain bounded by a number of circles) and more straightforward (in that it consists of a single explicit polynomial).

2. The Disc

We first consider the unit disc $\Delta = \{|z| \leqslant 1\}$ and then by simple transformations deduce results for a disc in a general position.

Lemma 2.1. *Let* $z_1, ..., z_N$ *be distinct points of* $\mathbb{C}$. *Define* $\phi_k(z, \zeta) = (z - \zeta)(z\bar{z}_k - \bar{\zeta} z_k)$ $(1 \leqslant k \leqslant N)$ *and put*

$$R_{k,N}(z) = \prod_{\substack{j=1 \\ j \neq k}}^{N} \left(\frac{\phi_k(z, z_j)}{\phi_k(z_k, z_j)} \right)^2,$$

$$W_k(z) = \left(\frac{1 + z\bar{z}_k}{1 + |z_k|} \right)^{m_k}.$$

where $m_k \in \mathbb{N}$ will be specified later, and set

$$P_k(z) = W_k(z)\, R_{k,N}(z).$$

Then, suppose that for some k, $z_k \in \partial \Delta$. Given $K > 0$, $\varepsilon > 0$, for all large m_k there is a $\delta > 0$ such that, if $\arg(z/z_k) = \theta$ $(0 \leqslant |\theta| \leqslant \pi)$, then

(a) $|P_k(z)| < 1 - K\theta^2$ $\qquad (|z| = 1, |\theta| < \delta)$,

(b) $|P_k(z)| < \varepsilon$ $\qquad (|z| = 1, \delta \leqslant |\theta| \leqslant \pi)$.

(c) *If $l \neq k$, $|P_l(z)| = O(\theta^2)$* $\qquad (|z| = 1, \theta \to 0)$ *uniformly with respect to the choice of m_l.*

Proof. If $z = z_k e^{i\theta}$, then

$$\begin{aligned} |\phi_k(z, \zeta)| &= |z_k e^{i\theta} - \zeta|\, |e^{i\theta} - \bar{\zeta} z_k| \\ &= |(e^{i\theta} - \zeta \bar{z}_k)(e^{i\theta} - \bar{\zeta} z_k)|. \end{aligned}$$

Since $\overline{\zeta \bar{z}_k} = \bar{\zeta} z_k$ one checks easily that

$$|R_{k,N}(z_k e^{i\theta})| = 1 + O(\theta^2) \qquad (\theta \to 0).$$

Also,

$$\begin{aligned} |W_k(z_k e^{i\theta})| = \left(\frac{|1 + e^{i\theta}|}{2}\right)^{m_k} &= \left|\cos \frac{\theta}{2}\right|^{m_k} \\ &= 1 - \frac{m_k \theta^2}{8} + O(\theta^4) \qquad (\theta \to 0). \end{aligned}$$

Hence, if $K > 0$ is given, there is a $\delta > 0$ so that (a) of Lemma 2.1 holds for all large m_k.

On the other hand, there is a κ $(0 < \kappa < 1)$ such that

$$|W_k(z_k e^{i\theta})| \leqslant \kappa^{m_k} \qquad (\delta \leqslant |\theta| \leqslant \pi),$$

where δ is that specified above. On $\{|z| = 1\}$ there is a bound for $|R_{k,N}(z)|$ depending only on $z_1, \ldots, z_N$ and, hence, if m_k is large enough both (a) and (b) are satisfied.

For $l \neq k$, there is a bound for the modulus of $P_l(z)$ with $\{\phi_l(z, z_k)/\phi_l(z_l, z_k)\}^2$ divided out that depends only on $z_1, \ldots, z_N$. Since the preceding factor has a double zero at $z = z_k$ the result (c) of Lemma 2.1 follows. ∎

THEOREM 2.2. *Let $z_1,\dots,z_n \in \partial\Delta$. In the preceding notation, if the exponents $m_1,\dots,m_n$ are suitably large and $\sigma_1,\dots,\sigma_n$ are any complex numbers of modulus at most 1 and $f(z)$ is the polynomial*

$$f(z)=\sum_{k=1}^{n}\sigma_k P_k(z),$$

then

$$\|f\|\leqslant 1,\qquad f(z_k)=\sigma_k\ (1\leqslant k\leqslant n),\qquad f(z_k)=0\ (n+1\leqslant k\leqslant N).$$

Proof. (a) and (c) of Lemma 2.1 cope with the behaviour of $f(z)$ on $\{|z|=1\}$ near to the points $z_1,\dots,z_n$ and (b) copes with the remainder of the unit circle. ∎

COROLLARY 2.2(i). *The result of the theorem holds for any disc $\{|z-\zeta|\leqslant\rho\}$, that is, given n points $w_1,\dots,w_n$ with $|w_k-\zeta|=\rho$ $(1\leqslant k\leqslant n)$ and $w_{n+1},\dots,w_N\in\mathbb{C}$; and numbers $\sigma_1,\dots,\sigma_n\in\mathbb{C}$ of modulus at most 1, then there is a polynomial $f(z)$ such that*

$$\|f\|=\max\{|f(z)|:|z-\zeta|\leqslant\rho\}\leqslant 1,$$
$$f(w_k)=\sigma_k\ (1\leqslant k\leqslant n),\qquad f(w_k)=0\ (n+1\leqslant k\leqslant N).$$

Proof. Let $z_k=(w_k-\zeta)/\rho$ $(1\leqslant k\leqslant N)$ and define

$$Q_k(z)=P_k\left(\frac{z-\zeta}{\rho}\right)$$

and set

$$f(z)=\sum_{k=1}^{n}\sigma_k Q_k(z).$$

The result of Corollary 2.2(i) now follows from Theorem 2.2. ∎

COROLLARY 2.2(ii). *The result of the theorem holds for any "disc exterior" $\{|z-\zeta|\geqslant\rho\}$, that is, given n points $w_1,\dots,w_n$ with $|w_k-\zeta|=\rho$ $(1\leqslant k\leqslant n)$ and $w_{n+1},\dots,w_N\in\mathbb{C}$; and numbers $\sigma_1,\dots,\sigma_n\in\mathbb{C}$ of modulus at most 1, then there is a function $f(z)$, which is a polynomial in $1/(z-\zeta)$, such that*

$$\|f\|=\max\{|f(z)|:|z-\zeta|\geqslant\rho\}\leqslant 1,\qquad f(w_k)=\sigma_k\ (1\leqslant k\leqslant n),$$
$$f(w_k)=0\ (n+1\leqslant k\leqslant N).$$

Proof. Let $z_k = \rho/(w_k - \zeta)$ $(1 \leqslant k \leqslant N)$ and define

$$S_k(z) = P_k\left(\frac{\rho}{z-\zeta}\right)$$

and set

$$f(z) = \sum_{k=1}^{n} \sigma_k S_k(z).$$

The result of Corollary 2.2(ii) now follows from Theorem 2.2. ∎

3. Finitely Connected Domains Bounded by Circles

The following results from Section 2:

COROLLARY 3.1. *Given $\sigma_k \in \mathbb{C}$, with $|\sigma_k| \leqslant 1$ $(k = 1,..., N)$. Let D be a finitely connected open domain bounded by circles, and let $w_1,..., w_N$ be given distinct points of ∂D. Then there exists $f(z)$ in $A(D)$ such that*

$$\|f\| \leqslant 1 \qquad \text{and} \qquad f(w_k) = \sigma_k \quad (k = 1,..., N).$$

Proof. Dealing with the general case involves rather cumbersome notation, and so we shall restrict our attention to doubly connected domains. From our discussion it should be immediately clear how the construction may be developed in the general case.

Let D be a doubly connected domain with outer boundary $\Gamma_0 = \{|z-\zeta_0| = \rho_0\}$ and inner boundary $\Gamma_1 = \{|z-\zeta_1| = \rho_1\}$. Suppose that $w_1,..., w_n \in \Gamma_0$ and $w_{n+1},..., w_N \in \Gamma_1$. Assume that $\sigma_1,..., \sigma_N \in \mathbb{C}$ and each is of modulus 1 at most. Define $Q_k(z)$ as in the proof of Corollary 2.2(i). Define $S_k(z)$ as in the proof of Corollary 2.2(ii) except that we consider the w_k in the order $w_{n+1},..., w_N, w_1,..., w_n$ and not in their natural order as we did in the earlier proof. Finally, set

$$f(z) = \sum_{k=1}^{n} \sigma_k Q_k(z) + \sum_{k=1}^{N-n} \sigma_{n+k} S_k(z).$$

Then $f(z)$ is analytic in D and by arguments similar to those in the proofs of Theorem 2.2 and its corollaries one finds that $|f(z)| \leqslant 1$ $(|z-\zeta_0| = \rho_0)$ and $|f(z)| \leqslant 1$ $(|z-\zeta_1| = \rho_1)$ and, hence, by the maximum modulus principle,

$$\|f\| \leqslant 1.$$

Also from the definition

$$f(w_k) = \sigma_k \qquad (1 \leqslant k \leqslant N),$$

and so the proof is complete. ∎

4. Finitely Connected Domains Bounded by Jordan Curves

Corollary 4.1. *Given $\sigma_k \in \mathbb{C}$ with $|\sigma_k| \leqslant 1$ $(k = 1,..., N)$. Let D be a finitely connected open domain bounded by Jordan curves, and let $\zeta_1,..., \zeta_N$ be given distinct points of ∂D. Then there exists a function $F(z)$ in $A(D)$ such that*

$$\|F\| \leqslant 1 \qquad \text{and} \qquad F(\zeta_k) = \sigma_k \quad (k = 1,..., N).$$

Proof. It is known that there is a mapping $\phi(z)$ which is a homeomorphism from $\bar{D}$ onto the closure of a domain of the kind considered in Corollary 3.1 and which is analytic in D. (See [5, Chap. IX, in particular, Theorems IX, 2, 35].)

Let $w_k = \phi(\zeta_k)$ and consider the function $f(z)$ of Section 3 relative to the w_k and σ_k. If $F(z) = f(\phi(z))$, then $F(z)$ is analytic in D, continuous on $\bar{D}$, $F(\zeta_k) = \sigma_k$ $(1 \leqslant k \leqslant N)$ and $\|F\| \leqslant 1$. ∎

5. Explicit Constructions in Special Cases

Explicit constructions of functions $f(z)$ in $A(D)$, satisfying

$$\|f\| \leqslant 1 \qquad \text{and} \qquad f(z_k) = \sigma_k \quad (|\sigma_k| \leqslant 1), \tag{7}$$

have so far only been given for a disc and a domain between two circles. We now give further constructions based on these.

(i) *Ellipse.* Suppose D is the elliptical region $|w + \sqrt{(w^2 - 1)}| < \rho$ $(\rho > 1)$, w_k are given points on ∂D, and σ_k are given in $\mathbb{C}$ $(|\sigma_k| \leqslant 1)$. Let $N = 2n$, $z_k = \rho^{-1}[w_k + \sqrt{(w_k^2 - 1)}]$, $z_{n+k} = z_k^{-1}\rho^{-2}$ $(k = 1,..., n)$, and define

$$f(w) = \sum_{k=1}^{n} \sigma_k\{P_k(\rho^{-1}[w + \sqrt{(w^2 - 1)}]) + P_k(\rho^{-1}[w - \sqrt{(w^2 - 1)}])\},$$

where P_k is as defined in Section 2. Then f has the properties (7). For $f(w) = \sum \sigma_k\{P_k(z) + P_k(z^{-1}\rho^{-2})\}$, where $\rho z = w + \sqrt{(w^2 - 1)}$, is an analytic function of $z + z^{-1}\rho^{-2}$ and, hence, of w. Also $f(w_k) = \sum \sigma_k\{P_k(z_k) + P_k(z_{n+k})\} = \sigma_k$ from Theorem 2.2.

(ii) *Eliptical ring*: *Ellipse with a slit.* The construction of Section 3 with $\zeta_0 = \zeta_1 = 0$ and $N = 2n$ leads immediately to a corresponding construction for the elliptical ring

$$D\colon \rho_0 < |w + \sqrt{(w^2 - 1)}| < \rho_1 \qquad (\rho_0 \geqslant 1),$$

under the conformal mapping

$$z = w + \sqrt{(w^2 - 1)}, \qquad z_k = w_k + \sqrt{(w_k^2 - 1)}.$$

For the function $F(w) = f[w + \sqrt{(w^2 - 1)}]$, where f is the function of Corollary 3.1, is in $A(D)$ and satisfies (7) (with f, z replaced by F, w).

In the case $\rho_0 = 1$, D degenerates into the region between a slit $|w + \sqrt{(w^2 - 1)}| = 1$ (i.e., $-1 \leqslant w \leqslant 1$, w real) and an ellipse, where we differentiate between the upper and lower edges of the slit in defining ∂D and the class $A(D)$.

6. Interpolation Projections

THEOREM 6.1. *For any finitely connected open domain D bounded by Jordan curves, the projection I_n from $A(D)$ onto polynomials of degree $n - 1$ which interpolate in n given points $\{z_k\}$ of ∂D, has Chebyshev norm*

$$\|I_n\| = \sup_z \sum_{k=1}^{n} |l_k(z)|,$$

where l_k are the fundamental polynomials (2) *in the Lagrange interpolation formula.*

Proof. This follows immediately from Corollary 4.1 and the discussion of Section 1. ∎

For multiply connected regions, polynomials in z are not really the most appropriate approximation subspaces, and so we now consider some important special cases.

THEOREM 6.2. *For the open doubly connected domain D between the circles $|z - \zeta_0| = \rho_0$ and $|z - \zeta_1| = \rho_1$ $(\rho_1 < \rho_0)$, the projection $I_{n,m}$ on polynomials of degree n in $(z - \zeta_1)^{-1}$ and $m - 1$ in $z - \zeta_0$ which interpolate in $n + m$ given points $\{z_k\}$ of ∂D, has Chebyshev norm*

$$\|I_{n,m}\| = \sup_z \sum_{k=1}^{n+m} |L_k(z)|,$$

where

$$L_k(z) = \left(\frac{z_k - \zeta_1}{z - \zeta_1}\right)^n l_k(z), \qquad k = 1, \dots, n + m. \tag{8}$$

Proof. Clearly $L_k(z)$ are the fundamental polynomials in the relevant Lagrange-type interpolation formula

$$(I_{n,m} f)(z) = \sum_{k=1}^{n+m} f(z_k) L_k(z).$$

The result, therefore, follows from Corollary 4.1 by extending the treatment of Section 1. (See [6], where this case is discussed in detail with numerical results.) ∎

COROLLARY 6.3. *For the open elliptical ring* $D\colon 1 \leqslant \rho_0 < |w + \sqrt{(w^2-1)}| < \rho_1$ *bounded by two ellipses with foci* ± 1, *or in the degenerate case* $\rho_0 = 1$, *for an ellipse D with a slit between its foci* ± 1, *the projection* I_n^* *from* $A(D)$ *onto functions of the form*

$$A_n(w) + \sqrt{(w^2-1)}\, B_{n-1}(w), \tag{9}$$

which interpolate in $2n+1$ *given points* $\{w_k\}$ *of* ∂D, *has Chebyshev norm*

$$\|I_n^*\| = \sup_w \sum_{k=1}^{2n+1} |L_k|w + \sqrt{(w^2-1)}||,$$

where L_k *is defined by* (8) *for* $\zeta_1 = 0$. *In the case* $\rho_0 = 1$, *we differentiate between the upper and lower edges of the slit in defining* ∂D *and the class* $A(D)$.

Proof. This result follows immediately from Theorem 6.2. by setting $\zeta_0 = \zeta_1 = 0$, $m = n + 1$, and identifying I_n^* with $I_{n,n+1}$ under the conformal mapping $z = w + \sqrt{(w^2-1)}$. [Since $w = \frac{1}{2}(z + z^{-1})$ and $\sqrt{(w^2-1)} = \frac{1}{2}(z - z^{-1})$, the form (9) corresponds to a polynomial of degrees n in both z^{-1} and z.] ∎

Note that Corollary 6.3, for the case $\rho_0 = 1$, is particularly relevant to applications in fracture mechanics. Indeed an approximation of a form equivalent to (9) is frequently used to represent a complex stress function for an elastic body having a straight crack between $w = \pm 1$. For example, Erdogan [7] uses (9) in differentiated form to obtain approximate solutions to crack problems.

7. Multivariate Functions

Clearly all the results in this paper may be extended to functions of several complex variables, and the reader is referred to [8] for a discussion of the details.

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